VERTEX-REINFORCED RANDOM WALK ON $\mathbb{Z}$ WITH SUB-SQUARE-ROOT WEIGHTS IS RECURRENT.

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Abstract. We prove that vertex-reinforced random walk on $\mathbb{Z}$ with weight of order k^α, for $\alpha \in [0, 1/2)$, is recurrent. This confirms a conjecture of Volkov for $\alpha < 1/2$. The conjecture for $\alpha \in [1/2, 1)$ remains open.

1. Introduction

Linearly vertex-reinforced random walk (VRRW for short), introduced by Pemantle [3], was studied on $\mathbb{Z}$ by Pemantle and Volkov [4]. A striking phenomenon was proved for this model [4, 7]: the random walk will eventually visit just 5 sites on $\mathbb{Z}$ almost surely.

In contrast, Volkov later in [8] studied non-linear vertex reinforced random walk on $\mathbb{Z}$ with some weight function $w : \{0, 1, 2, \dots\} \to (0, \infty)$. This process, denoted by $(X_n, n \geq 0)$ is defined as follows. Fix $X_0 = 0$. Then for all $n \geq 0$,

$$\mathbb{P}(X_{n+1} = X_n \pm 1 | X_1, \dots, X_n) = \frac{w(Z_n(X_n \pm 1))}{w(Z_n(X_n - 1)) + w(Z_n(X_n + 1))}, \qquad (1)$$

where $Z_n(y) = \#\{m \leq n : X_m = y\}$ is the local time in $y \in \mathbb{Z}$ at time n. For $w_k = k^\alpha(c + o(1))$, $\alpha \geq 0$, Volkov proved the existence of phase transition for this model. That is, there is a large time T_0 such that after T_0, the walk visits $2, 5$ or ∞ sites when $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$ respectively. In the case of $\alpha < 1$, though it was proved that the random walk will visit infinitely many sites, it is not clear whether it will visit every site of $\mathbb{Z}$ infinitely many times with probability 1. Namely, the question whether the random walk is recurrent[1] was left open.

Recently, Schapira was able to move one step further towards a positive answer to this question, and in [5] proved a 0-1 law for VRRW on $\mathbb{Z}$ with weight of order k^α, for $\alpha \in [0, 1/2)$. In this paper, we show that in this regime the walk is in fact recurrent.

Theorem. *Vertex reinforced random walk on $\mathbb{Z}$ with weight $w(k) \approx k^\alpha$, $\alpha \in [0, 1/2)$, is recurrent.*

The notation $w(k) \approx k^\alpha$ means that the ratio between the two quantities is bounded between two constants independent of k (except for $w(0)$ on which we make no requirements).

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[1]The definition of recurrence we use here is that the random walk visits every vertex of $\mathbb{Z}$ infinitely many times almost surely and the definition of transience is that the random walk visits every vertex of $\mathbb{Z}$ finitely many times almost surely.

The proof of the theorem consists of a martingale argument, which is a modification of a similar martingale argument used in [2] for edge-reinforced random walk on $\mathbb{Z}$. Another ingredient of the proof is the fact that for small α, the random walk will not visit the nearby sites too many times before moving to a new site (see Lemma 2). This fact was basicly proved by Schapira via a kind of domino principle. This is the part of the proof that only works for $0 \le \alpha < 1/2$.

Let us remark that Arvind Singh [6] arrived at a similar result simultaneously, using a martingale argument similar in spirit but different in some technical details.

2. Proof

Recall the definition of $Z_i(j)$. For all $i \in \mathbb{N}$, we define a sequence of random variables $F_i : \mathbb{Z} \setminus \{0\} \to \mathbb{R}^+$

$$F_i(v) = \begin{cases} \sum_{j=0}^{v-1} \dfrac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} & \text{if } v > 0; \\ \sum_{j=v}^{-1} \dfrac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} & \text{if } v < 0. \end{cases} \tag{2}$$

Note that $F_i(\cdot)$ depends on the history of the random walk up to time i and is $\mathscr{F}_i-$measurable where $\mathscr{F}_i$ is the σ-field spanned by $X_1, \dots, X_i$. Then we have the following lemma.

Lemma 1. *Let $X_0 = 0$. Let $T = \min\{i > 0 : X_i = 0\}$ i.e. the first time the process returns to the origin. Then $\{F_{\min(T,i)}(X_{\min(T,i)}) : i = 1, 2, \dots\}$ is a supermartingale.*

Proof. We think about moving from $F_i(X_i)$ to $F_{i+1}(X_{i+1})$ as being composed of two steps: moving X and updating the weights. We will prove that F_i satisfies the following two properties:

(i) *harmonicity:* for all $i \in \mathbb{N}$, with respect to the random walk's transition probability at time i, $F_i(v)$ is harmonic (in v) on $\mathbb{Z} \setminus \{0\}$. In other words, the first step is a martingale.

(ii) *monotonicity:* for any fixed $v \in \mathbb{Z} \setminus \{0\}$, $F_i(v)$ is monotone decreasing in i.

Let us prove (i). We condition on $\mathscr{F}_i$, and denote $v = X_i$ for brevity, and assume $v > 0$ (the other case is similar). We get

$$\begin{aligned}
\mathbb{E}(F_i(X_{i+1}) \,|\, \mathscr{F}_i) &= \mathbb{P}(X_{i+1} = v+1) F_i(v+1) + \mathbb{P}(X_{i+1} = v-1) F_i(v-1) \\
&= \frac{w(Z_i(v+1))}{w(Z_i(v-1)) + w(Z_i(v+1))} \sum_{j=0}^{v} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} \\
&\quad + \frac{w(Z_i(v-1))}{w(Z_i(v-1)) + w(Z_i(v+1))} \sum_{j=0}^{v-2} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} \\
&= \sum_{j=0}^{v-2} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} + \frac{w(Z_i(v+1))}{w(Z_i(v-1)) + w(Z_i(v+1))} \cdot \\
&\quad \cdot \left(\frac{1}{w(Z_i(v-1)) \cdot w(Z_i(v))} + \frac{1}{w(Z_i(v)) \cdot w(Z_i(v+1))} \right) \\
&= \sum_{j=0}^{v-1} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} = F_i(v).
\end{aligned}$$

Hence, we proved (i).

(ii) follows from the fact that for fixed j, $Z_i(j)$, the random walk's local time is monotone increasing in time i.

Now by harmonicity and monotonicity of $F_i(v)$, one has

$$\mathbb{E}(F_{i+1}(X_{i+1}) \,|\, \mathscr{F}_i) \leq \mathbb{E}(F_i(X_{i+1}) \,|\, \mathscr{F}_i) = F_i(X_i),$$

so $F_i(X_i)$ is a supermartingale. □

Remark. Lemma 1 holds more generally for any vertex reinforced random walk on $\mathbb{Z}$ with increasing weight sequence. In fact it holds for any self-interacting process where the vertex weights are increasing, and $\mathbb{Z}$ may be replaced with any tree (also remarked in [1]).

To prove the theorem we need a second lemma. Let T_n denote the hitting time of a vertex $n \in \mathbb{Z}$. Then,

Lemma 2. *Almost surely,* $I := \liminf_{n\to\infty} Z_{T_n}(n-1) < \infty$.

Proof. The claim is equivalent to showing

$$\lim_{k\to\infty} \mathbb{P}\left(\liminf_{n\to\infty} Z_{T_n}(n-1) > k\right) = 0. \tag{3}$$

Note that for any fixed k

$$\begin{aligned}\mathbb{P}\left(\liminf_{n\to\infty} Z_{T_n}(n-1) > k\right) &= \mathbb{P}\left(\cup_{N\geq 0} \cap_{n\geq N} \{Z_{T_n}(n-1) > k\}\right) \\ &= \sup_{N\geq 0} \mathbb{P}\left(\cap_{n\geq N}\{Z_{T_n}(n-1) > k\}\right) \\ &\leq \sup_{N\geq 0} \mathbb{P}\left(Z_{T_N}(N-1) > k\right).\end{aligned}$$

We now apply formula (4.3) in [5], which claims that

$$\sup_{N\geq 0} \mathbb{P}\left(Z_{T_N}(N-1) > k\right) \leq Ce^{-ck^c}, \tag{4}$$

where c and C are some positive constants (possibly depending on the weight w). Hence, (3) follows from (4). This concludes the proof of the lemma. □

By the same argument as the proof of Lemma 2, one can prove the same behaviour in the negative direction i.e. $\liminf_{n\to-\infty} Z_{T_n}(n+1) < \infty$.

Finally, we also use the 0-1 law proved by Schapira, which is stated as follows.

Lemma 3. [5, Theorem 1.1] *Vertex-reinforced random walk on $\mathbb{Z}$ with weight $w(k) \approx k^\alpha$, $k \geq 1$ for some $\alpha \in [0, 1/2)$, is either recurrent or transient.*

Proof of the theorem. By Lemma 3, we know that X_n is either recurrent or transient. Now suppose X_n is transient, then X_n will visit the origin just finitely many times almost surely. By Lemma 1, $F_i(X_i)$ will be a supermartingale eventually. Since it is positive, it converges to a finite random variable almost surely. On the other hand, by Lemma 2 there will be infinitely many vertices N, such that the increment of $F_i(X_i)$ at time T_N is bounded from below by a positive random variable. Indeed, the only update to Z that happens at time T_N is the increasing of $Z(N)$ to 1, but $Z(N)$ does not appear in the sum defining F_{T_n-1}. Hence

$$F_{T_n}(X_{T_n}) - F_{T_n-1}(X_{T_n-1}) = \frac{1}{w(Z_{T_N}(N-1))w(1)} \geq \frac{1}{w(I)w(1)} > 0$$

(where I is still $\liminf_{n\to\infty} Z_{T_n}(n-1) < \infty$). This contradicts the convergence of $F_i(X_i)$. Therefore, we can conclude the theorem. □

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